

Schwarzschild Black Hole Quantum Statistics from $Z(2)$ Orientation Degrees of Freedom and its Relations to Ising Droplet Nucleation

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Abstract

Generalizing previous quantum gravity results for Schwarzschild black holes from 4 to $D \geq 4$ space-time dimensions yields an energy spectrum $E_n = \alpha n^{(D-3)/(D-2)} E_{P,D}$, $n = 1, 2, \dots$, $\alpha = O(1)$, where $E_{P,D}$ is the Planck energy in that space-time. This energy spectrum means that the quantized area $A_{D-2}(n)$ of the $D-2$ dimensional horizon has universally the form $A_{D-2} = n a_{D-2}$, where a_{D-2} is essentially the $(D-2)$ th power of the D -dimensional Planck length. Assuming that the basic area quantum has a $Z(2)$ -degeneracy according to its two possible orientation degrees of freedom implies a degeneracy $d_n = 2^n$ for the n -th level. The energy spectrum with such a degeneracy leads to a *quantum* canonical partition function which is the same as the *classical* grand canonical potential of a primitive Ising droplet nucleation model for 1st-order phase transitions in $D-2$ spatial dimensions. The analogy to this model suggests that E_n represents the surface energy of a "bubble" of n horizon area quanta. Exploiting the well-known properties of the so-called critical droplets of that model immediately leads to the Hawking temperature and the Bekenstein-Hawking entropy of Schwarzschild black holes. The values of temperature and entropy appear closely related to the imaginary part of the partition function which describes metastable states.

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1 Introduction

In previous papers [1, 2, 3] I discussed the quantum statistics of the energy spectrum

$$E_n = \alpha \sqrt{n} E_P, \quad n = 1, 2, \dots, \quad E_P = c^2 \sqrt{c \hbar / G}, \quad \alpha = O(1), \quad (1)$$

with degeneracies

$$d_n = g^n, \quad g > 1. \quad (2)$$

Many authors have proposed that Schwarzschild black holes in 4-dimensional spacetimes have such a spectrum (see the list in Ref. [1]).

The (quantum) canonical partition function of the above spectrum not only leads to the Hawking temperature and the associated Bekenstein-Hawking entropy, but, very amazingly, it is formally the same as the *classical* grand canonical potential of the primitive droplet nucleation model in the context of first-order phase transitions in *two* space dimensions. Thus the so-called "holographic principle" [4, 5, 6], namely that the essential physics of black holes should be associated with the 2-dimensional horizon, is very evident here and comes out as a result!

Furthermore, as the canonical partition function of the spectrum (1) becomes complex for the degeneracies (2), because $g > 1$, one leaves the well-established framework of equilibrium thermodynamics (KMS states) and moves on the perhaps more slippery ground of metastable states and nonequilibrium thermodynamics [7].

The paper is organized as follows: I first briefly summarize postulates and arguments from previous papers [1, 2, 3, 8] generalizing the spectrum (1) to $D \geq 4$ space-time dimensions [3].

Furthermore, I shall argue in chapter 3 that the much debated degeneracy (2) may be attributed to the $Z(2)$ degrees of freedom associated with the two possible orientations of the basic area quantum, i.e. $g = 2$! This is the main new idea of the present paper, namely that the huge entropy of black holes is related to the $Z(2)$ -valued geometrical quantity "orientation".

Moreover, it will be shown that for all $D \geq 4$ the resulting *quantum* canonical partition function is the same as the *classical* grand canonical potential of the (nonrelativistic) primitive droplet nucleation model in $D - 2$ spatial dimensions, too. The essential features of the droplet nucleation model needed for our purpose are discussed in chapter 4. From a comparison of the two models one sees that the energy E_n in (1) and its D -dimensional

generalizations are to be interpreted as the surface energies of "bubbles" of n area quanta.

Using known results from the droplet nucleation model - especially the notion of a critical droplet - the Hawking temperature and the Bekenstein-Hawking entropy are derived in chapter 5, up to a normalization factor which is discussed separately. In chapter 6 it is pointed out how an effective Hamiltonian of the Born-Infeld type can be used to describe the spectrum plus the degeneracy factor. Its classical mechanical counterpart has some amusing properties, too. Here it is interesting that the imaginary part of the corresponding classical partition function is the same as that of the quantum mechanically one. The real parts are different, however. Chapters 4-6 rely strongly on Ref. [3].

2 The quantum Schwarzschild BH spectrum in D -dimensional space-time

The spectrum (1), already discussed in Refs. [1, 2, 3], may be argued for heuristically as follows [8]: A canonical Dirac-type treatment of spherically symmetric pure Einstein gravity leads to a reduced 2-dimensional phase space having only the ADM mass M and a canonically conjugate time functional T as (observable) pair of variables [9].

In this simple model an observer at spatial (flat) infinity will only have the mass M and his own proper time τ available in order to describe the system. His very simple Schrödinger equation for it is

$$i\hbar\partial_\tau\phi(\tau) = Mc^2\phi(\tau) , \quad (3)$$

which has the plane wave solutions

$$\phi(M, \tau) = \chi(M)e^{-\frac{i}{\hbar}Mc^2\tau} , \quad (4)$$

where $M > 0$ is assumed. If the system with mass M stays there forever, then M is to be considered as a continuous quantity. However – just like in the case of plane waves representing a system confined to a 1-dimensional spatial interval with finite extension L –, if the above system, represented by the plane wave (4), has only a finite duration Δ then this property may

be (crudely) implemented by imposing periodic boundary conditions on the plane wave (4), implying the relation [8]

$$c^2 M \Delta = 2\pi\hbar n, \quad n = 1, 2, \dots \quad (5)$$

As to the boundary condition (5) it is worthwhile to stress the following point: The assumption that the wave function (4) has the period Δ does not mean that the asymptotic time τ is periodic. It merely means that the system is in a (quasi-) stationary state (4) during the time interval Δ . This is completely analogous to a system of free particles in a finite spatial interval of length L where they are described by a plane wave with periodic spatial boundary conditions. Such a property of the wave function does not mean that space itself is confined to an interval or periodic.

The question now is how to choose Δ ! As the only intrinsic quantity available at spatial infinity to characterize a time interval is M itself, or a function of it, namely the Schwarzschild radius $R_S(M)$, we assume [8]

$$\Delta = \gamma R_S(M)/c, \quad (6)$$

where γ is a dimensionless number of order 1. The choice (6) means that we deal with the (quasistationary) "formation" period of the black hole [8], not with the Hawking evaporation period which, according to the law of Stefan-Boltzmann has the different time scale $\propto R_S^{D-1}$. Details will be discussed at the end of chapter 5.

Inserting the ansatz (6) into relation (5) gives the mass quantization condition

$$\gamma c M_n R_S(M_n) = 2\pi\hbar n, \quad n = 1, 2, \dots \quad (7)$$

For $D = 4$ we have $R_S = 2MG/c^2$ and the spectrum (1) results. However, one can assume the relation (7) to be valid in any dimension $D \geq 4$, because the Schrödinger Eq. (3) has to hold in any such dimension [11]!

The relation (7) is also an appropriate generalization of a Bohr-Sommerfeld type quantization of the 2-dimensional horizon as suggested very early by Bekenstein [12], Mukhanov [13] (see also Bekenstein's recent review [14]) and – in the context of string theory – by Kogan [15]:

If we interpret cM as canonical momentum and R_S as canonical coordinate, then the relation (7) is, qualitatively, nothing else but the old-fashioned quantum counting of phase space cells.

Those readers who are - at present - skeptical about the arguments leading to the Eqs. (5) and (6) may take them as mere assumptions which -

together with the degeneracies discussed below - lead to interesting physical consequences including those generally expected from black holes. The close correspondence to the droplet nucleation model to be discussed below will shed additional light on the physical significance of those assumptions.

In the following it is convenient to use these notations: We put $D = 1 + d = 2 + \tilde{d}$: d gives the number of space dimensions and \tilde{d} the spatial dimensions of the black hole horizon.

In D space-time dimensions the Schwarzschild radius is given by [16]

$$R_S(M) = \left(\frac{16\pi G_D M}{c^2 \omega_{\tilde{d}} \tilde{d}} \right)^{1/(\tilde{d}-1)}, \quad (8)$$

where G_D is the gravitational constant in D -dimensional space-time and

$$\omega_{\tilde{d}} = 2\pi^{(\tilde{d}+1)/2} / \Gamma((\tilde{d}+1)/2)$$

is the volume of $S^{\tilde{d}}$. Inserting this R_S into the relation (7) gives the energy spectrum

$$\begin{aligned} E_n = M_n c^2 &= \alpha_D n^{1-\eta} E_{P,D}, \quad \eta = 1/\tilde{d}, \\ \alpha_D &= \left(\frac{(2\pi)^{\tilde{d}-2} \omega_{\tilde{d}} \tilde{d}}{8 \gamma^{\tilde{d}-1}} \right)^{\eta} \approx O(1), \end{aligned} \quad (9)$$

where

$$E_{P,D} = \left(\frac{\hbar^{D-3} c^{D+1}}{G_D} \right)^{1/(D-2)}, \quad l_{P,D} = \left(\frac{\hbar G_D}{c^3} \right)^{1/(D-2)} \quad (10)$$

are the corresponding Planck energy and Planck length in D space-time dimensions, respectively.

Combining the Eqs. (7) and (8) yields the following important expression for the \tilde{d} -dimensional quantized area of the horizon

$$A_{\tilde{d}}(n) = (R_S(M_n))^{\tilde{d}} \omega_{\tilde{d}} = n a_{D-2}, \quad a_{D-2} = \frac{32 \pi^2}{\gamma^{\tilde{d}}} l_{P,D}^{\tilde{d}}, \quad n = 1, 2, \dots \quad (11)$$

Thus, the horizon is built up additively and equidistantly from Planck-sized area elements a_{D-2} universally for all space-time dimensions $D \geq 4$!

Notice that

$$A_{\tilde{d}}(n_1 + n_2) = A_{\tilde{d}}(n_1) + A_{\tilde{d}}(n_2),$$

but (Minkowski's inequality)

$$E_{n_1+n_2} < E_{n_1} + E_{n_2} \text{ for } n_1, n_2 \geq 1 .$$

Therefore it is energetically advantageous to form *one* large area instead of several smaller ones, the energy difference being "radiated" away! Thus, the following picture emerges (see also chapter 5): If two spherically symmetric horizon quantum "bubbles", characterized by the numbers n_1 and n_2 , merge, then the area of the resulting quantum bubble is the sum of the area of the original surfaces whereas the resulting surface energy is lower than the sum of the original ones. This is a kind of quantum version of Hawking's classical area theorem [10].

The spectra (11) and (9), respectively, can be substantiated considerably by an appropriate group theoretical quantization [17] of the classical system associated with the symplectic form $d\tau \wedge dM$ (which is symplectically equivalent to $d\varphi \wedge dp$, $\varphi \in (-\pi, +\pi]$, $p > 0$) in terms of the group $SO^\uparrow(1, 2)$ and the positive discrete series of its unitary representations [11].

The main result of this group theoretical quantization is that

$$A_{\tilde{d}}(k; \tilde{n}) \propto k + \tilde{n} , \tilde{n} = 0, 1, 2, \dots , \quad (12)$$

where $k = 1, 2, \dots$ characterizes the irreducible unitary representation of $SO^\uparrow(1, 2)$. Physically k determines the energy value of the ground state for which $\tilde{n} = 0$. The unitary representation corresponding to the spectrum (11) belongs to $k = 1$ and can be realised by complex-valued functions $f(z)$, $|z = x + iy| = \sqrt{x^2 + y^2} < 1$, with the scalar product

$$(f_1, f_2) = \frac{1}{\pi} \int_{|z|<1} dx dy \bar{f}_1(z) f_2(z) \quad (13)$$

and the orthonormal basis

$$\phi_{\tilde{n}}(z) = \sqrt{\tilde{n} + 1} z^{\tilde{n}} , \tilde{n} = 0, 1, 2, \dots . \quad (14)$$

The operator with the eigenvalues $n = \tilde{n} + 1$ and the eigenfunctions $\phi_{\tilde{n}}$ is $zd/dz + 1$. For more details see Ref. [11]. Notice that the eigenfunctions $\phi_{\tilde{n}}$, $\tilde{n} > 1$, are – up to the normalization – just powers of ϕ_1 ! This is reminiscent of a Fock space structure: the \tilde{n} -area quantum wave function is essentially the \tilde{n} -th power of the 1-area quantum wave function!

It is an interesting and encouraging result [18] that for $D = 4$ the (horizon) area operator in the spherically symmetric sector of loop quantum gravity has eigenvalues which for large n are proportional to n , too.

3 Degeneracy of the spectrum due to the $Z(2)$ -freedom of orientation

I would like to stress that a relationship like (2) for the degeneracies is very important for the thermodynamics involved [1, 2, 3]! Like the spectrum (1) the ansatz (2) for the degeneracies has a longer history, too:

It is already implicitly contained in Bekenstein's early work (see his review [19]). Afterwards Zurek and Thorne [20] interpreted the degeneracies as the number of quantum mechanically distinct ways the black hole could be made by infalling quanta. If their total number is n then the associated combinatorial sum

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

yields the degeneracy (2) with $g = 2$. Similarly, Mukhanov suggested [13] to take as d_n the number of possible ways to build up the n -th level if the ground state is nondegenerate. This gives $d_n = 2^{n-1}$. 't Hooft [21, 6] pointed out that one gets the right relation between the area of the horizon and the entropy if one divides the horizon into Planck-sized cells and attributes two degrees of freedom (i.e. a $Z(2)$ -symmetry) to each cell. Sorkin and others [22, 23] [24] have argued similarly. See also the recent review by Bekenstein [14].

The big question then is, *where does such a degeneracy in terms of a microscopic $Z(2)$ -symmetry come from geometrically?* If it is already present for a pure Einstein quantum gravity system – uncoupled to matter – it has to have a geometrical origin!

Actually there is such a geometrical degree of freedom, namely *orientation*: Most of the physically interesting manifolds are orientable (see the Appendix), having exactly two equivalence classes $[\omega_{\pm}^n = f_{\pm}(x)dx^1 \wedge \dots \wedge dx^n, f_{+}(x) > 0, f_{-}(x) < 0] \equiv \sigma_M = +1, -1$ of everywhere nonvanishing n -forms $\in \Lambda^n(M)$ of a n -dimensional manifold M . The convention is that the R^n with a natural ordering of its basis has $\sigma_{R^n} = \sigma[e^1 \wedge \dots \wedge e^n] = +1$ and that this orientation is induced on M by the orientation-preserving atlas of coordinate systems.

Orientation is a global property of a classical manifold, and – similar to space reflection parity in atomic and particle physics – it appears suggestive to attribute this degree of freedom to the smallest area quanta of the corresponding quantum gravity system, too. Here the classical system is the

horizon bifurcation sphere Σ^{D-2} which is orientable (see Appendix). So, if we associate with each area quantum a_{D-2} in Eq. (11) a $Z(2)$ -degree of freedom $\sigma = \pm 1$ then a state with n such quanta has a degeneracy 2^n . Because of the properties of "orientation" this degeneracy is universally the same for all dimensions D .

As the two orientational degrees of freedom of the elementary area quanta appear to be energetically undistinguishable – at least in a first approximation – they lead to the immense degeneracy (2) implying the Bekenstein entropy which is very much larger than that associated with the mere two orientation degrees of freedom of a classical geometrical sphere.

The two properties – the quantum statistical and the classical one – may be related as follows: Suppose there are – in a second approximation – possible new microscopic (weak) attractive interactions between the elementary orientation degrees of freedom, perhaps similar to the exchange interactions between spins due to Fermi-Dirac statistics, or more likely, similar to "attractions" due to Bose-Einstein statistics (with condensation).

Actually, the variable $\sigma(i)$ of the i -th horizon area quantum is an Ising-type variable and there is the question whether it shares other properties of the Ising model [25]:

At high (Hawking) temperatures (black holes with small masses) the microscopic Ising variables form a ("para"-) phase of non-correlated Ising "spins".

But then there may be something like "spontaneous orientation" at very low temperatures! Recall that for the 2-dimensional Ising model on a square lattice we have the relation $J/k_B T_c \approx 0.4$ between the critical temperature T_c and the coupling J of nearest neighbors. As the Hawking temperature T_H of macroscopic black holes ($D = 4$) with masses of about 10 solar masses is of the order $10^{-8} K \approx 10^{-12} eV$, then the energy scale J is extremely small *if* the corresponding Ising T_c is of an order similar to that of T_H .

The possible occurrence of a spontaneous orientation at very low temperatures of very large black holes could explain why we can speak of a macroscopic classical horizon the associated sphere of which has just two possible orientations corresponding to the classical limit $\hbar \rightarrow 0, T_H \rightarrow 0$.

In any case it is very important to clarify the problem whether the geometrical degree of freedom "orientation" is merely a kinematical one or whether it has dynamical properties, too, and how those are related to Einstein's gravity.

What could correspond to the external magnetic field in our case?

Classically a suitable representative of the equivalence class σ which is invariant under orientation-preserving diffeomorphisms is the volume element $\sqrt{\gamma^{(\tilde{d})}}(u) du^1 \wedge \dots \wedge du^{\tilde{d}}$ of $\Sigma^{\tilde{d}}$, where $\gamma^{(\tilde{d})}$ is the determinant of the induced metric on $\Sigma^{\tilde{d}}$. Diffeomorphism-invariant *external* fields are the curvature scalar ${}^{\tilde{d}}R$, a (cosmological) constant or any other classical diffeomorphism invariant function, which even might be related to matter fields, e.g. $F^{jk}F_{jk}(u)$. The total contribution of all volume elements is then given by the integral over $\Sigma^{\tilde{d}}$. Thus, if, e.g. the external field is a constant and if $\Sigma^{\tilde{d}}$ is a sphere, then the total "interaction" is proportional to the area of $\Sigma^{\tilde{d}}$! The situation is similar to the coupling of p -branes to $(p+1)$ -forms in the context of theories of strings and p -branes [26, 27].

Quantum theoretically the above volume elements should be replaced by area quanta "coupled" to some classical surface and the integral replaced by a corresponding sum. In order to work this out one has to know more about the "interactions" of the orientation degrees of freedom, similar to the interactions of atomic spins (magnetic moments) in para- and ferromagnets.

Taking the Ising orientation degrees of freedom $\sigma(i) = \pm 1$ into account implies that the wave function (14) is to be replaced by

$$\phi_{\tilde{n}}(z) \chi(\sigma(0), \sigma(1), \dots, \sigma(\tilde{n})) , \quad \tilde{n} = 0, 1, \dots,$$

where χ is symmetric with respect to the exchange of any pair of the $\sigma(i)$, $i = 1, \dots, \tilde{n}$.

These few remarks indicate that introducing the Ising spin type variable "orientation" as a new degree of freedom opens a wide range of new theoretical possibilities for analyzing black holes and other gravitational systems. Here we shall merely exploit the fact that the spectrum (9) corresponds to surface energies of droplets of n Ising spins in a background of opposite Ising spins (see the next chapter).

Another point to comment on is the free parameter γ . We shall see below how the Gibbons-Hawking geometrical approach [28] to the partition function may be used to restrict it, but it is worth mentioning that such a free parameter also occurs in loop quantum gravity [29], in the discussion of the Schwarzschild black hole thermodynamics of Matrix theory [30] and in the AdS/CFT correspondence [31].

In a microcanonical approach to the thermodynamic properties of the

system we have for the entropy

$$S(n)/k_B = \ln d_n = n \ln 2 = \frac{\ln 2}{a_{D-2}} A_{D-2}(n). \quad (15)$$

If we define, in analogy to the classical expression $T = \partial U / \partial S$, U : internal energy, for sufficiently large n ,

$$T \approx \frac{E_{n+1} - E_n}{S(n+1) - S(n)} \quad (16)$$

we get

$$T \propto n^{-\eta} \quad (17)$$

which corresponds to the classical relationship $T \propto 1/R_S(M)$ [16].

The quantum canonical partition function resulting from the degeneracy (2) and the spectrum (9) may be written as

$$\begin{aligned} Z_D(t, x) &= \sum_{n=0}^{\infty} e^{nt - n^{1-\eta}x}, \\ t &= \ln g, \quad \eta = 1/\tilde{d} \equiv 1/(D-2), \\ x &= \beta \alpha_D E_{P,D}, \quad \beta = \frac{1}{k_B T}. \end{aligned} \quad (18)$$

It is important for the following discussion to keep g arbitrary and put $g = 2$ only in some final results. One reason is that the series (18) does not converge for $g > 1$ and one has to make analytical continuations. In the following it is mathematically convenient to start the sum (18) at $n = 0$ instead of $n = 1$.

The function $Z_D(t, x)$ obeys the linear PDE

$$\partial_t^{\tilde{d}-1} Z_D = (-1)^{\tilde{d}} \partial_x^{\tilde{d}} Z_D, \quad (19)$$

which can be used to determine Z_D in closed form. This has been done [32] for $\tilde{d} = 2$ and 3.

4 Primitive droplet nucleation model in $D - 2$ space dimensions

The model assumption that 1st-order phase transitions are initiated by the formation (homogeneous nucleation) of expanding droplets of the new phase

within the old phase is a popular and important one (see the reviews [33] with their references to the original literature).

In its most primitive form the droplets are assumed to be spherical and to consist of n constituents (e.g. droplets of Ising spins on a lattice or liquid droplets of molecules etc.), the "excess" energy ϵ_n of which is given by a "bulk" term proportional to the volume n and a term proportional to the surface $n^{1-\eta}$, $\eta = 1/\tilde{d}$, where $\tilde{d} \geq 2$ is the spatial dimension of the system:

$$\epsilon_n = -\hat{\mu} n + \phi n^{1-\eta} , \quad \eta = 1/\tilde{d} . \quad (20)$$

In the case of negative Ising spin droplets, formed in a background of positive spins by turning an external magnetic field H slowly negative - below the critical temperature -, the coefficient $\hat{\mu}$ in Eq. (20) takes the form $\hat{\mu} = -2H$ and for liquid droplets of n molecules condensing from a super-saturated vapour one has $\hat{\mu} = \mu - \mu_c$, where μ is the chemical potential and μ_c its critical value at condensation point. ϕ is the constant surface tension. Assuming the average number $\bar{\nu}(n)$ of droplets with n constituents to be proportional to a Boltzmann factor,

$$\bar{\nu}(n) \propto e^{-\beta \epsilon_n}, \quad \beta = \frac{1}{k_B T} , \quad (21)$$

and that the droplets form a non-interacting dilute gas leads to the grand canonical potential $\psi_{\tilde{d}}$ per spin or per volume

$$\psi_{\tilde{d}}(\beta, t = \beta \hat{\mu}) = \ln Z_G = p\beta = \sum_{n=0}^{\infty} e^{tn - xn^{1-\eta}} , \quad (22)$$

$$t = \beta \hat{\mu}, \quad x = \beta \phi ; \quad p : \text{pressure}, \quad d\psi_{\tilde{d}} = -U d\beta + \bar{n} dt . \quad (23)$$

(Again: for physical reasons the sum (22) may not start at $n = 0$ but at some finite $n_0 > 0$. This can easily be taken care of.)

Obviously the sum (22) is the same as in Eq. (18)! The correspondence suggests to interpret the energy spectrum (9) as representing some kind of surface energy associated with a quantized horizon forming a *bubble* of n area quanta, the Planck energy $E_{P,D}$ playing the role of a surface tension. The presence of this energy guarantees the additive and equidistant quantum surface building property (11).

Notice a qualitative difference between the droplet nucleation picture and the corresponding black hole horizon formation: The \tilde{d} -dimensional droplets

have a $(\tilde{d}-1)$ -dimensional boundary where the surface tension can be "felt", whereas the horizon bifurcation sphere is a closed \tilde{d} -dimensional surface, a "bubble" so to speak, held together by its surface tension!

Analogously, the "orientation degeneracy field" $t = \ln g$ appears as a "driving property", similar to an external magnetic field.

It is worth mentioning that the above primitive Ising nucleation model with its purely spherical droplets is itself only a simplified version of the genuine Ising model which can describe metastable states itself [34] and which can allow for non-spherical excitations, too.

The interpretation of the sum (22) as a grand canonical potential comes about as follows [35, 36, 2]:

One starts with a canonical partition function Z_c , where the same terms as above are summed up. However, then one has the thermodynamics of single n -droplets, but there may be N of them. If one assumes these to be non-interacting and indistinguishable, then the grand canonical partition function is

$$Z_G = \sum_N \frac{Z_c^N}{N!} = e^{Z_c}, \quad (24)$$

which leads to the grand canonical potential $\psi_{\tilde{d}}$ of Eq. (22).

It will be important, however, that we interpret Z_D of Eq. (18) as a *canonical* partition function, where $g = e^t$ describes the fixed, *temperature-independent*, degeneracies of the corresponding quantum levels, whereas in the droplet nucleation model $z = e^t, t = \hat{\mu}\beta$, is the *temperature-dependent* fugacity of a classical Boltzmann gas!

Notice that $\psi_{\tilde{d}}$ contains no explicit information about properties of the phases before and after the phase transition. Consider, e.g. a vapour \rightarrow fluid phase transition. Then the properties of the vapour are only very indirectly present in $\psi_{\tilde{d}}$, namely in form of the surface energy ϕ of the droplets emerged in the vapour. The model here merely is supposed (for more details see the Refs. [33]) to describe that (metastable!) part of the Van der Waals isotherm in the (V, p) -plane which starts where, with decreasing volume, the (theoretically!) strict equilibrium line of the Maxwell construction branches off to the left, till the (local) maximum of the Van der Waals "loop", the "spinodal" point, is reached.

The series (22) converges for $t \leq 0$ only. This follows, e.g. from the Maclaurin-Cauchy integral criterium [37]. In applications to metastable systems, however, one is interested in the behaviour of $\psi_{\tilde{d}}(t, x)$ for $t \geq 0$. This

calls for an analytic continuation in t or in the fugacity $z = e^t$ which reveals a branch cut of $\psi_{\tilde{d}}$ from $z = 1$ to $z = \infty$ [38].

Qualitatively the following happens: For $t < 0$ (i.e. positive magnetic field) ϵ_n increases monotonically with n , making the corresponding terms in $\psi_{\tilde{d}}$ decrease monotonically. The small droplets are favoured and no phase transition occurs.

If, however, $t > 0$, then ϵ_n has a maximum for

$$n^* = \left(\frac{(1-\eta)\phi}{\hat{\mu}} \right)^{\tilde{d}} = \left(\frac{(1-\eta)x}{t} \right)^{\tilde{d}}, \quad x = \phi\beta, \quad (25)$$

with

$$\epsilon^* \equiv \epsilon_{n^*} = a\eta(1-\eta)^{\tilde{d}-1}, \quad a = \frac{\phi^{\tilde{d}}}{\hat{\mu}^{\tilde{d}-1}} = \frac{x^{\tilde{d}}}{\beta t^{\tilde{d}-1}}, \quad (26)$$

after which ϵ_n becomes increasingly negative with increasing n and the series (22) explodes!

The physical interpretation is the following: If, by an appropriate fluctuation, a "critical droplet" of "size" $n > n^*$ has appeared, it is energetically favored to grow. Such an over-critical droplet – and others of a similarly large size – will destabilize the original phase and will send the system to the phase for which it has served as a nucleus!

The energy ϵ^* may be interpreted as a measure for the critical barrier of the free energy the system has to "climb" over in order to leave the metastable state for a more stable one.

Furthermore, the rate Γ for the transition of the metastable state to the more stable one is proportional to $\exp(-\beta\epsilon^*)$. However, calculating the rate is no longer a problem of *equilibrium* thermodynamics. One has to deal with tools for *non-equilibrium* processes like the Fokker-Planck equation etc. For the droplet model this was essentially done by Becker and Döring [39]. They assumed a stationary situation, where a steady flow of small, but in size increasing, droplets leave the metastable state and all over-critical droplets which have passed the barrier are removed from the system.

Their approach was considerably improved by Langer [40] who related the transition rate Γ to the imaginary part of $\psi_{\tilde{d}}$. This can be seen roughly as follows: If one turns the sum $\psi_{\tilde{d}}$ in Eq. (22) into an integral by interpreting n as a continuous variable:

$$\tilde{\psi}_{\tilde{d}} = \int_0^\infty dn e^{\beta(\hat{\mu}n - \phi n^{1-\eta})}, \quad (27)$$

then a saddle point approximation [35, 32] for large β gives the asymptotic expansion

$$\tilde{\psi}_{\tilde{d}} \sim (1 - \eta)^{\tilde{d}/2} \sqrt{\frac{\pi \tilde{d}}{2\hat{\mu}\beta}} \left(\frac{\phi}{\hat{\mu}}\right)^{\tilde{d}/2} e^{-\beta a \eta (1 - \eta)^{\tilde{d}-1}} (i + O(1/\beta)) . \quad (28)$$

Here the path in the complex n -plane goes from $n = 0$ to n^* and then parallel to the imaginary axis to $+i\infty$ [32]. (Thus, only half of the associated Gaussian integral along the steepest descents contributes!)

The crucial point is that the saddle point is given by n^* and the associated ϵ^* of Eqs. (25) and (26), that is to say, by the critical droplet!

The leading term in the saddle point approximation (28) is purely imaginary. Performing a Fokker-Planck type analysis, Langer found [40] that the transition rate Γ is essentially proportional to the imaginary part $\Im(\tilde{\psi}_{\tilde{d}})$, the other factor being a "dynamical" one, genuinely related to non-equilibrium properties. For further discussions see the reviews mentioned in Ref. [33].

An essential point for us here is the result that the imaginary part of $\psi_{\tilde{d}}$ can be interpreted, at lest intuitively, in terms of equilibrium concepts although it is related to non-equilibrium properties which are, however, near to stationary situations.

5 Hawking temperature and Bekenstein-Hawking entropy

We are now ready to apply the droplet nucleation model results to the Schwarzschild black hole: If we denote the "critical" term in the series (18) by Z_D^* , we have

$$Z_D^* = e^{-[\eta(1 - \eta)^{\tilde{d}-1} x^{\tilde{d}}]/t^{\tilde{d}-1}} . \quad (29)$$

The essential point now is that $t = \ln g$ here is no longer a temperature dependent quantity – as in the droplet model – but a fixed number. Therefore the equation of state for the associated internal energy,

$$U^* = -\frac{\partial \ln Z_D^*}{\partial \beta} = (1 - \eta)^{\tilde{d}-1} \left(\frac{x}{t}\right)^{\tilde{d}-1} \sigma_D E_{P,D} = \tilde{d} \epsilon^* , \quad (30)$$

can be used to determine the (inverse) temperature β^* needed for a (potential) heat bath, if the rest energy U^* is given! Solving Eq. (30) for x and

using the relation (8) between Schwarzschild radius R_S and mass $M = U^*/c^2$ we obtain

$$\beta^* = \lambda \left(\frac{4\pi R_S^*}{(\tilde{d}-1)\hbar c} \right) , \quad \lambda \equiv \frac{t \tilde{d} \gamma}{8\pi^2} , \quad (31)$$

where $R_S^* = R_S(M = U^*/c^2)$ (Eq. (8)).

The expression in the bracket of Eq. (31) is exactly the inverse Hawking temperature in D -dimensional space-time [16], if we identify $U^* = Mc^2$, where M is the macroscopic rest mass of the black hole! Thus, up to a numerical factor λ of order 1, we obtain the Hawking temperature in this way.

For the entropy $S_D^* = \beta^* U^* + \ln Z_D^*$ we get

$$\begin{aligned} S_D^*/k_B &= (1-\eta)^{\tilde{d}} (x^*)^{\tilde{d}} / t^{\tilde{d}-1} = (1-\eta) \beta^* U^* \\ &= t n^* = \ln(g^{n^*}) , \quad x^* = \beta^* \sigma_D E_{P,D} , \end{aligned} \quad (32)$$

where n^* is the same as in Eq. (25).

If we express S_D^* in terms of the \tilde{d} -dimensional surface $A_{\tilde{d}} = \omega_{\tilde{d}} (R_S)^{\tilde{d}}$, we have

$$S_D^*/k_B = \lambda \frac{A_{\tilde{d}}}{4l_{P,D}^{\tilde{d}}} = \lambda \frac{c^3 A_{\tilde{d}}}{4\hbar G_D} . \quad (33)$$

So, up to the same numerical factor λ we already encountered in connection with the inverse temperature, we obtain the Bekenstein-Hawking entropy!

The mean square fluctuations of the energy

$$(\Delta E)^2 = \partial_{\beta}^2 (\ln Z_D^*) = - \frac{(1-\eta)^{\tilde{d}-1} (\tilde{d}-1)}{t^{\tilde{d}-1}} x^{\tilde{d}-2} (\sigma_D E_{P,D})^2 \quad (34)$$

are negative (negative specific heat!), but relatively small for large masses because

$$\frac{(\Delta E)^2}{U^*} = - \frac{\tilde{d}-1}{\beta^*} , \quad (35)$$

which appears to be quite a universal relation: the r.h.s. of Eq. (35) depends only on \tilde{d} and β^* !

As the factor γ , up to now, is a free parameter, we can possibly choose it in such a way that the above prefactor λ equals one:

$$\gamma = \frac{8\pi^2}{t \tilde{d}} . \quad (36)$$

A suitable argument for such a normalization $\lambda = 1$ comes from the practically classical geometrical result $S/k_B = A_{\tilde{d}}/(4l_{P,D}^{\tilde{d}})$ of Gibbons and Hawking [28], derived from the euclidean section of the Schwarzschild solution. In view of the surprising approximate equalities of the classical and the quantum theoretical values for S , one may use the classical result for normalizing the quantum one.

There is a corresponding analogue in QED, where the physical value of the electric charge e in the quantum theory (to all orders) is normalized via the universal classical total Thomson cross section $\sigma_{tot} = (8/3)\pi r_0^2$, $r_0 = e^2/(mc^2)$, for Compton scattering in the limit of vanishing photon energy [41].

There is a very similar universal low energy absorption cross section for spherically symmetric black holes [42], namely the classical area of the horizon, which serves the same purpose as the Gibbons-Hawking normalization mentioned above.

One has to be careful here, however, because only Z_D^* has the same simple exponential form as one finds in the Gibbons-Hawking approach in lowest order. Fluctuations lead to prefactors with powers of x in front of Z_D^* as can already be seen if we take the imaginary part $Z_{i,D}$ of the saddle point approximation (28) as a slightly more sophisticated "pseudo" partition function, or, if we take the imaginary part of the purely imaginary partition function for the (euclidean) Schwarzschild black hole if one includes "quadratic" quantum fluctuations around the classical solution [43, 44, 2]. Such additional powers of x lead to corrections to β and to logarithmic corrections of the entropy (33) [1, 2].

As to possible connections between statistically defined entropies and the ones obtained geometrically and as to possible quantum corrections see the recent review by Frolov und Fursaev [45].

The corresponding normalization problem in loop quantum gravity has been discussed in Refs. [29].

The normalization (36) leads to the expression

$$a_{D-2} = 4 (\ln g) l_{P,D}^{\tilde{d}} \quad (37)$$

for the area quantum of Eq. (11).

I said "pseudo" partition function because it implies negative mean square fluctuations, see Eq. (34) and Refs. [1, 2], a property which appears to be associated with the metastability of the system. More theoretical work seems

to be necessary in order to understand these partially surprising features better beyond the realm of strict equilibrium thermodynamics [7].

Altogether we see the following picture emerging in the framework of the approach advocated here:

The formation (nucleation) of a Schwarzschild black hole means quantum mechanically that the horizon consists of a (huge) sum of Planck-sized area elements each of which carries two Ising ($= Z(2)$)-like degrees of freedom related to the two possible orientations. These additional degrees of freedom lead to the Bekenstein-Hawking entropy and to the associated Hawking temperature. The general picture is not new - see the Refs. in ch. 3 above -, new is the suggestion that the Ising degrees of freedom are associated with the geometrical property "orientation" and that one should take the Ising model seriously here because of the geometrical $Z(2)$ freedom of orientation.

This suggestion should, of course, apply to other black holes as well. Take, for instance, the Reissner-Nordström one, where the value of the entropy for the extreme limit ($Q = M$) is controversial: There are arguments for a vanishing entropy [46, 47, 48, 49, 50, 51] and against [52, 53], especially in string theory which relies heavily on the extremal property [54, 55, 56]. At present I have only a very tentative comment because the quantum mechanics of Reissner-Nordström black holes is still in a preliminary state [57, 58, 51, 59, 60, 61, 53].

Consider the submanifold characterized by $r_- = M - \sqrt{M^2 - Q^2} \leq r \leq r_+ = M + \sqrt{M^2 - Q^2}$ in the non-extreme case. The boundaries at $r = r_-$, r_+ are oriented in such a way that the normal vectors point outward relative to the manifold at both boundaries, i.e. they point in opposite directions (see the Appendix). If we now take the limit $r_- \rightarrow r_+$ ($Q \rightarrow M$) then we have the singular coincidence of two spheres with opposite orientations. As the temperature $T = (r_+^2 - Q^2)/(4\pi r_+^3)$ vanishes in the limit $Q \rightarrow M$, one may be tempted to expect $S \rightarrow 0$ if there is "spontaneous orientation" at $T = 0$!

"Orientation" is classically a global geometrical property, which means that it is related to the "topology" of the geometry in question. The importance of this aspect for the entropy of black holes has been stressed in a number of recent papers [46, 47, 48, 49, 50, 62, 63, 64, 65].

It is remarkable that the *quantum* statistics of Schwarzschild black holes is formally the same as that of the *classical* primitive Ising droplet nucleation model in $D - 2$ space dimensions, represented as a classical grand canonical ensemble. This shows how the holographic principle [4, 5] is implemented here. It further indicates how the quantum "background" is hidden behind a

classically appearing facade, very probably formed by the thermal physics of the horizon, which is being built up during the nucleation of the black hole.

The "blurring" of the quantum properties is also indicated by the fact that the traces of the Bose statistics of the quanta (1) are rather hidden, contrary to, e.g., the canonical partition function of the simple harmonic oscillator:

If we look at the exact expression for Z_4 in closed form, derived in Ref. [1],

$$\begin{aligned} Z_4(t, x) &= \int_0^\infty d\tau \hat{K}(\tau, x) \frac{1}{1 - e^{(t - \tau)}} , \\ \hat{K}(\tau, x) &= \frac{x}{2\sqrt{\pi\tau^3}} e^{-x^2/(4\tau)} , \\ \Re[Z_4(t, x)] &= \text{p.v.} \int_0^\infty d\tau \hat{K}(\tau, x) \frac{1}{1 - e^{t - \tau}} , \quad \Im[Z_4(t, x)] = \pi \hat{K}(t, x) , \end{aligned} \tag{38}$$

only the factor $1/(e^{t-\tau} - 1)$ in the principal value integral for the *real* part of Z_4 indicates Bose statistics, whereas the - for our discussion above crucial - imaginary part does not show such traces (see also the next chapter). This property may shed new light on the information loss problem [66].

The role of the external magnetic field (or a corresponding chemical potential) in the droplet case finds its correspondence in the degeneracy factor g which intuitively represents the "driving property" leading to the formation of the black hole by nucleation.

What is really new, compared to the droplet model, is that the free energy barrier ϵ^* determines its own temperature, namely T_H , and the associated entropy (32), as a function of the total internal energy $U^* = Mc^2$. This is a genuinely quantum mechanical effect, because $t = \ln g$ is a fixed number, not a temperature-dependent quantity as in the droplet nucleation case. Here lies the real difference. This temperature, after nucleation, then becomes that of the black hole itself. Any thermal radiation emitted from the horizon carries the imprint of this temperature.

There is an interesting similarity to the Bose-Einstein condensation, where the critical temperature is determined as a function of the particle number density at vanishing chemical potential [67].

In the above physical interpretation of the nucleation process concerning the black hole I followed the droplet nucleation picture and have assumed that the black hole is the *result* of the decay of metastable states to a more stable

one (the black hole) which then Hawking radiates with the corresponding temperature.

Instead one might think of another interpretation, namely that the black hole itself is the metastable state which slowly decays, due to its interaction with the heat bath consisting of Hawking radiation. However, here the associated Stefan-Boltzmann evaporation time scale τ is – for $D = 4$ – proportional to R_S^3 (see, e.g., Ref. [68]), not proportional to R_S as assumed in Eq. (6). Generalizing the Stefan-Boltzmann law to D space-time dimensions yields $\tau \propto R_S^{D-1}$ accordingly. Thus, the "evaporation" interpretation is not appropriate here.

The nucleation of black holes has been discussed quite early by Gross, Perry and Yaffe in terms of the euclidean Schwarzschild instanton [44, 2].

One can arrive at similar results for the Hawking temperature and the Bekenstein-Hawking entropy as above if one performs a microcanonical counting of states (see Ref. [69] and the Eqs. (15)-(17) above), however, then one loses the very inspiring connection to the droplet nucleation picture.

6 Effective quantum and classical Hamiltonians

The partition function Z_D of Eq. (18) may be rewritten as

$$Z_D = \text{tr}(e^{-\beta \hat{H}}) , \quad \hat{H} = -\mu a^+ a + \epsilon (a^+ a)^{1-\eta} , \quad \mu = t/\beta , \quad \epsilon = \alpha_D E_{P,D} , \quad (39)$$

where a and a^+ are the annihilation and creation operators of the harmonic oscillator. If u_n is an eigenfunction of the harmonic oscillator, then $a^+ a u_n = n u_n$ and the assertion (38) follows immediately. As the trace is independent of the basis one uses for its calculation, one might also use another one, e.g. the coherent states $|z\rangle$, which are eigenstates of a with complex eigenvalues z . Doing so [70] for $\tilde{d} = 2$ leads to the same exact result as in Ref. [1].

It is interesting to look [71] briefly at the corresponding classical effective system: Let us define the classical quantity

$$\tilde{N} = \frac{1}{\hbar \omega_0} \left(\frac{1}{2m} p^2 + \frac{m}{2} \omega_0^2 q^2 \right) . \quad (40)$$

After an appropriate rescaling of q and p we have the effective Hamiltonian

$$\tilde{H} = -\mu \tilde{N} + \epsilon \tilde{N}^{1-\eta} , \quad \tilde{N} = \frac{1}{2} (p^2 + q^2) , \quad (41)$$

leading to

$$\dot{p} = -\frac{\partial \tilde{H}}{\partial q} = -\tilde{\omega} q, \quad \tilde{\omega}(\tilde{N}) = -\mu + (1-\eta)\epsilon\tilde{N}^{-\eta}, \quad (42)$$

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} = \tilde{\omega} p. \quad (43)$$

It follows that \tilde{N} is a constant of motion \tilde{N}_0 for the associated fictitious point particle and that this particle moves on a circle with radius $\sqrt{2\tilde{N}_0}$ in phase space with frequency $\tilde{\omega}$ which is a function of \tilde{N}_0 (this is a new feature compared to the usual harmonic oscillator).

The critical value $\tilde{\omega} = 0$ results if $\tilde{N} = \tilde{N}_c = ((1-\eta)\epsilon/\mu)^{\tilde{d}}$ which is just the same as n^* from Eq. (25) above and for which \tilde{H} has its maximum. For $\tilde{N} < \tilde{N}_c$ the frequency $\tilde{\omega}$ is positive and for $\tilde{N} > \tilde{N}_c$ it is negative.

As

$$\dot{q} = ((1-\eta)\epsilon\tilde{N}^{-\eta} - \mu)p, \quad (44)$$

it is in general not at all trivial to calculate the Lagrange function $L(q, \dot{q})$, because one has to solve an algebraic equation if one wants $p(\dot{q})$ from Eq. (44). Already for $\tilde{d} = 2$ this equation is of order 4. If $\mu = 0$ we get in this case

$$L(q, \dot{q}) = -q\left(\frac{1}{2}\epsilon^2 - \dot{q}^2\right)^{1/2}, \quad (45)$$

which may be interpreted as a simple example of a Born-Infeld type Lagrangean [72].

Finally I mention how the classical partition function Z_{cl} associated with the Hamiltonian (41) looks like:

$$Z_{cl} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp dq}{2\pi\hbar} \exp\left\{-\beta\left[-\frac{\mu}{2\hbar}(p^2 + q^2) + \frac{\epsilon}{\sqrt{2\hbar}}(p^2 + q^2)^{1/2}\right]\right\}. \quad (46)$$

Introducing polar coordinates in the (q, p) -plane and making appropriate substitutions yields

$$Z_{cl} = 2 \int_0^{\infty} du u e^{tu^2 - xu}, \quad t = \beta\mu, \quad x = \epsilon\beta. \quad (47)$$

The integral exists for $t < 0$ and gives [73]

$$Z_{cl} = -\frac{e^{-x^2/(8t)}}{t} D_{-2}\left(\frac{x}{\sqrt{-2t}}\right), \quad (48)$$

where $D_p(z)$ is the parabolic cylinder function of order p . Continuing now from negative to positive t finally yields

$$Z_{cl} = -\frac{1}{t}\Phi(1, 1/2; -\frac{x^2}{4t}) + i\frac{\sqrt{\pi}x}{2t^{3/2}}e^{-x^2/(4t)} , \quad (49)$$

where $\Phi(a, c; z)$ is the confluent hypergeometric function [74] with $\Phi(a, c; z = 0) = 1$.

What is remarkable here is that the *imaginary* part of the classical partition function Z_{cl} is the same as that of the quantum theoretical one (38). The real parts are different (put $x = 0$), however, reflecting the difference between classical and quantum mechanics. This shows again that in the context of the quantum statistics of black holes *dissipative* properties of the system are important [3, 75]! The "invariance" of the imaginary part under quantization reminds one of the Rutherford scattering cross section for charged particles which is the same in mechanics and quantum mechanics, due to the long range of the Coulomb forces.

7 Conclusions

The considerations above reveal a coherent picture of the quantum theory of isolated Schwarzschild black holes and the associated quantum statistics:

The first basic feature is that the quantum horizon area spectrum is additive and equidistant – like the spectrum of the harmonic oscillator – and this universally so in any space-time dimension $D \geq 4$, the basic quantum being of the order of $l_{P,D}^{D-2}$. The associated energies E_n are to be interpreted as the surface energies of a droplet of n elementary quanta.

As the classical horizon is geometrically an orientable sphere with two possible orientations (in any dimension!) it appears natural to assign this $Z(2)$ degree of freedom to each single area quantum separately. This leads to a degeneracy $d_n = 2^n$ of the n -th level.

The associated quantum mechanical states consist of vectors of one of the Hilbert spaces for (positive discrete series) irreducible unitary representations of the group $SO^\dagger(1, 2)$ or its covering groups [11], the n -quanta (area) states being tensorially multiplied by a wave function for n Ising variables representing a configuration of the two possible orientations of each of the n area quanta.

The resulting canonical quantum statistics is formally the same as that of the classical grand canonical statistics of the Ising nucleation model in first-order phase transitions for metastable states: When the droplet of n area quanta reaches a critical size the Schwarzschild system turns into a new phase, that of a black hole.

The imaginary part of the complex partition functions yields Hawking's relation between temperature and macroscopic mass of the black hole and Bekenstein's relation between its entropy and the macroscopic size of its horizon. The associated normalization problem for the Hawking temperature and the Bekenstein-Hawking entropy may be solved by referring to the essentially classical euclidean partition function approach of Gibbons and Hawking.

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Appendix

Geometrical orientation of manifolds

This appendix is intended to point out some mathematical literature dealing with the notion of "orientation" of a manifold and to indicate some of its properties. A neat introduction into the subject is given in chapters 6 and 7 of Ref. [76], see also chapter 3 of Ref. [77], chapter 5 of Ref. [78], chapter 4 of Ref. [79], chapters 15 and 17 of Ref. [80], chapters 9 and 10 of Ref. [81]. As to more advanced topics like the relationship between orientation and characteristic classes (Stiefel-Whitney and Euler) see Refs. [82] and [83]. For the purpose of the present paper it is important that n -dimensional spheres, $n \geq 1$, are orientable [84]. As the Kruskal-Schwarzschild and the extended Reissner-Nordström manifolds may be covered by one coordinate patch [85] on which $\det(g_{\mu\nu})$ does not vanish anywhere, they are orientable, too.

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